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AN ANALOGUE OF THE PHRAGMEN-LINDELOF PRINCIPLE FOR SOLUTIONS OF NON-UNIFORMLY DEGENERATE ELLIPTIC EQUATIONS OF THE SECOND ORDER

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Abstract. We consider a second-order elliptic equation in divergence form that is non-uniformly degenerating at infinity. Conditions on the coefficients of the equation are found that ensure the validity of the Phragmén-Lindelöf principle for its solutions. This paper investigates the qualitative properties of solutions to second-order elliptic equations characterized by non-uniform degeneracy. We establish an analogue of the Phragmén–Lindelöf principle, asserting that the asymptotic behavior of solutions at infinity is strictly governed by the local characteristics of the ellipticity degeneracy. By employing estimates for the Green's functions and analyzing the capacitative potential of the underlying sets, we derive structural conditions on the weight functions $w(x)$. It is shown that for solutions whose growth is below a critical exponential order, the maximum principle remains valid in unbounded domains. Furthermore, the results demonstrate that non-uniformity in the degeneracy leads to a significant narrowing of the uniqueness class compared to the classical uniformly elliptic case.

Keywords: Non-uniform degeneracy, elliptic equation, capacitative potential, Phragmen-Lindelof principle, Greens function.

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1. Introduction

Let E_n be an n-dimensional euclidean space of points $x = (x_1, \dots, x_n), n \geq 3$ D is an unbounded domain in E_n with boundary $\partial\Omega \in C^2$. Let us consider the equation in D

$$Lu(x) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) = 0 \tag{1}$$

assuming that for $x \in D$ and $\xi \in E_n$ the condition holds

$$\gamma\cdot\sum_{i=1}^{n}\lambda_i(x)\xi_i^2\le\sum_{i,j=1}^{n}a_{ij}(x)\xi_i\xi_j\le\gamma^{-1}\cdot\sum_{i=1}^{n}\lambda_i(x)\xi_i^2. \qquad (2)$$

Here $\gamma\in(0,1]$ is a constant, $\left\|a_{ij}(x,t)\right\|$ is a real symmetric matrix,

$$\lambda_i(x)=\left(\frac{\omega_i^{-1}(R+\rho(x))}{R+\rho(x)}\right)^2, i=1,\dots,n, \rho(x)=\sum_{i=1}^{n}\omega_i(|x_i|).$$

Herewith $\omega_i(t), i=1,\dots,n$ are strictly monotonically increasing functions for $t\in[0,\infty)$ as $t\to+\infty, \omega_i(t)\to+\infty, \omega_i^{-1}(t)$ are functions inverse to $\omega_i(t)$ and there exist $p>1, q>n$ and $A>0$ such that for sufficiently large R and for $i=1,\dots,n$, the following holds:

$$p\cdot\omega_i(R)\le\omega_i(2R)\le 2\cdot\omega_i(R), \qquad (3)$$

$$\left(\frac{\omega_i^{-1}(R)}{R}\right)^{q-1}\cdot\int_0^{\omega_i^{-1}(R)}\left(\frac{\omega_i(\tau)}{\tau}\right)^q d\tau\le AR. \qquad (4)$$

The question of the behavior of solutions of second-order elliptic equations in unbounded domains has been studied by many authors. In this work, we apply the growth lemma to prove theorems of the Fragmen-Lindelöf type. In the case of uniformly elliptic equations, a proof of a similar fact can be found in [7, 8, 9, 10]. Regarding equations with uniform degeneration, the works [5, 6, 4] can be noted. We also point to the works [11, 3, 2] for a one class of elliptic equations with non-uniform power degeneration at a point.

Conditions in some notations and definitions. Let $W_{2,\lambda}^1(D)$ denote the Banach space of functions $u(x)$ defined on D for which the norm

$$\|u\|_{W_{2,\lambda}^1(D)}=\left(\int_D\left(u^2+\sum_{i=1}^{n}\lambda_i(x)u_i^2\right)dx\right)^{1/2}$$

is finite, and let $\overset{\circ}{W}{}_{2,\lambda}^1(D)$ denote the completion $C_0^\infty(\overline{D})$ with respect to the norm of the space $W_{2,\lambda}^1(D)$, where $u_i=\frac{\partial u}{\partial x}, \lambda=\left(\lambda_1(x),\dots,\lambda_n(x)\right)$. Denote by $\overset{\circ}{W}{}_{p,\lambda}^1(D)$ the conjugate space of $W_{p`,\lambda}^{-1}(D)$, where $p>1$ and $\frac{1}{p}+\frac{1}{p`}=1$. It is known that $W_{p`,\lambda}^{-1}(D)$ consists of generalized functions defined in $D$, which are the first derivatives of functions from $L_{p`}(D)$. By $M_0(D)$ we denote the space of functions satisfying the uniform Lipschitz condition in D and vanishing near ∂D.

Definition 1. We say that a function $u(x)\in\overset{\circ}{W}{}_{2,\lambda}^1(D)$ is nonnegative on a set E in the $L_p(D)$ sense (or almost everywhere) if

$$\operatorname{mes}\{[x: u(x)<0]\cap E\}=0.$$

Definition 2. We say that a function $u(x) \in \overset{\circ}{W}{}^1_{2,\lambda}(D)$ is nonnegative on a set E in the $L_p(D)$ sense (or almost everywhere) if there exists a sequence of functions $u_m(x) \in M_0(D)$ such that $u_m(x) \geq 0$ on E and

$$\lim_{m\to\infty} \|u_m - u\|_{W^1_{2,\lambda}(D)} = 0.$$

Obviously, these two definitions are completely different: the first is meaningless for a set E of measure zero, while the second may be meaningful for such a set E. It is easy to see that if a function u belongs to the space $\overset{\circ}{W}{}^1_{p,\lambda}(D)$ and is non-negative on the set E in the sense of $\overset{\circ}{W}{}^1_{p,\lambda}(D)$, then it is non-negative almost everywhere on the set E. If a function $u(x)$ belongs to the space $\overset{\circ}{W}{}^1_{p,\lambda}(D)$ and is non-negative almost everywhere on the set E, then it is non-negative in the sense of $\overset{\circ}{W}{}^1_{p,\lambda}(D)$ on any subset E that is a positive distance from ∂E.

The first property follows from the well-known theorem on the uniform convergence of a sequence of functions that converges in $L_p(D)$. The second property follows from the theorem on approximation by mean functions of elements $\overset{\circ}{W}{}^1_{p,\lambda}(D)$. By analogy, other inequalities can be defined in terms of $\overset{\circ}{W}{}^1_{p,\lambda}(D)$, for example $u(x) \geq const, u(x) \leq 0$ and $u(x) \geq v(x)$ on E.

Let $u(x) \in L_p(D), p \in (1, \infty), k > 0$ be a number. The function

$$\{u(x)\}^k = \begin{cases} u(x) & if \quad u(x) \leq k, \\ k & if \quad u(x) > k, \end{cases}$$

is called the k truncation of the function $u(x)$.

We will give a few simple statements.

Lemma 1. If $u(x) \in \overset{\circ}{W}{}^1_{p,\lambda}(D)$ and $k > 0$, then $\{u(x)\}^k \in \overset{\circ}{W}{}^1_{p,\lambda}(D)$ and if $u(x) \geq 1$ on E in the $\overset{\circ}{W}{}^1_{p,\lambda}(D)$ sense, then $\{u(x)\}^k = 1$ in the same sense.

Lemma 2. The set of functions $u(x) \in \overset{\circ}{W}{}^1_{2,\lambda}(D)$ such that $u(x) \geq 1$ on E in the sense of $\overset{\circ}{W}{}^1_{2,\lambda}(D)$ is a convex closed set.

Definition 3. Let us say that the function $u(x) \in W^1_{2,\lambda}(D)$ is non-negative on ∂D in the sense of $W^1_{2,\lambda}(D)$ if there exists a sequence of functions $u_m(x) \in C^1(\bar{D})$ such that $u_m(x) \geq 0$ on ∂D and

$$\lim_{m\to\infty} \|u_m - u\|_{W^1_{2,\lambda}(D)} = 0.$$

Lemma 3. Let $u(x) \in W^1_{p,\lambda}(D)$ and $u(x) \leq 0$ in ∂D in the sense of $W^1_{2,\lambda}(D)$. Then, for any $k > 0$, $\left(u(x) - \{u(x)\}^k\right) \in \overset{\circ}{W}{}^1_{p,\lambda}(D)$.

Definition 4. Let $f(x) \in L_2(D)$ be a given function and $\varphi(x) \in W^1_{2,\lambda}(D)$. The function $u(x) \in W^1_{2,\lambda}(D)$ is called a generalized solution of equation

$$Lu = f(x), \tag{5}$$

in D if for any $v(x) \in \overset{\circ}{W}{}^1_{2,\lambda}(D)$ the integral identity

$$\int_D \sum_{i,j=1}^n a_{ij}(x) u_i v_j \, dx = -\int_D \sum_{i,j=1}^n f(x) \cdot v(x) \, dx \tag{6}$$

holds.

If, in addition,

$$u(x) - \varphi(x) \in \overset{\circ}{W}{}^1_{2,\lambda}(D), \tag{7}$$

then the function u(x)u(x)u(x) is called a generalized solution of the first boundary value problem (5), (7).

Definition 5. A function $u(x) \in W^{1,loc}_{2,\lambda}(D)$ is called a generalized local solution of equation (1) in D if for every function $v(x) \in C^1(D)$ with compact support in D the integral identity

$$\int_D \sum_{i,j=1}^n a_{ij}(x) u_i v_j \, dx = 0 \tag{8}$$

holds.

Definition 6. A function $u(x) \in W^1_{2,\lambda}(D)$ is called L-subelliptic in D if for every nonnegative function $u(x) \in \overset{\circ}{W}{}^1_{2,\lambda}(D)$ the inequality

$$\int_D \sum_{i,j=1}^n a_{ij}(x) u_i v_j \, dx \leq 0$$

holds.

If $u(x)$ is L-subelliptic in D, then the function $-u(x)$ is called L-superelliptic in D.

Let us denote $R > 0, k > 0$ and $x^0 \in E_n$ by $E_R^{x^0}(k)$ -ellipsoid

$$\left\{ x \colon \sum_{i=1}^n \frac{\left(x_i - x_i^0\right)^2}{\left(\omega_i^{-1}(R)\right)^2} < k^2 \right\}$$

$\Pi_{R:k}(x^0)$-parallelepiped . The notation means that the positive constant c depends only on the contents of the brackets. Let, furthermore, $\frac{1}{2} \le \rho < \rho` \le 1$, $\bar{E}_R^{x^0}(\rho`) \subset D$, $x^0 \in E_R^0\left(\frac{1}{4\sqrt{n}}\right)$.

Lemma 4. For any generalized local solution $u(x)$ of equation (1) in D, the following holds:

$$\int_{E_R^{x^0}(\rho)} \sum_{i=1}^{n} \lambda_i(x) u_i^2 \, dx \le \frac{c_1(\gamma,\lambda,n)}{(\rho` - \rho)^2 \cdot R^2} \cdot \int_{E_R^{x^0}(\rho`)} u^2 dx. \tag{9}$$

Proof. Let $\psi(x) \in C_0^\infty\left(E_R^{x^0}(\rho`)\right), 0 \le \psi(x) \le 1$, such that $\psi(x) = 1$ in $E_R^{x^0}(\rho)$ and

$$|\psi_i| \le \frac{c_2(n)}{(\rho` - \rho) \cdot \omega_i^{-1}(R)}, \quad i = 1, \dots, n. \tag{10}$$

Set $v(x) = u(x) \cdot \psi^2(x)$ in the integral identity (8). We obtain:

$$\int_{E_R^{x^0}(\rho`)} \sum_{i,j=1}^{n} a_{ij}(x) u_i u_j \, \psi^2 dx = -2 \int_{E_R^{x^0}(\rho`)} u\psi \cdot \sum_{i,j=1}^{n} a_{ij}(x) u_i \psi_j \, dx \le$$

$$\le 2 \int_{E_R^{x^0}(\rho`)} |u| \cdot \psi \cdot \left(\sum_{i,j=1}^{n} a_{ij} u_i u_j\right)^{\frac{1}{2}} \cdot \left(\sum_{i,j=1}^{n} a_{ij}(x) \psi_i \psi_j\right)^{\frac{1}{2}} dx.$$

Now, using which holds for any $\varepsilon > 0$ the inequality $2ab \le \varepsilon \cdot a^2 + \frac{1}{\varepsilon} \cdot b^2$ with $\varepsilon = \frac{1}{2}$, we conclude:

$$\int_{E_R^{x^0}(\rho`)} \psi^2 \cdot \sum_{i,j=1}^{n} a_{ij}(x) u_i u_j \, dx = \frac{1}{2} \int_{E_R^{x^0}(\rho`)} \psi^2 \cdot \sum_{i,j=1}^{n} a_{ij}(x) u_i u_j \, dx +$$

$$+2 \int_{E_R^{x^0}(\rho`)} u^2 \cdot \sum_{i,j=1}^{n} a_{ij}(x) \psi_i \psi_j \, dx.$$

Taking into account conditions (2), we obtain:

$$\int_{E_R^{x^0}(\rho)} \sum_{i,j=1}^{n} \lambda_i(x) u_i^2 \, dx \le 4\gamma^{-2} \cdot \int_{E_R^{x^0}(\rho`)} u^2 \cdot \sum_{i=1}^{n} \lambda_i(x) \psi_i^2 \, dx. \tag{11}$$

If $x \in E_R^{x^0}(\rho`)$, then for $i = 1, \dots n$, $\left|x_i - x_i^0\right| < \rho` \cdot \omega_i^{-1}(R) \le \omega_i^{-1}(R)$, i.e. $|x_i| \le \left|x_i^0\right| + \left|x_i - x_i^0\right| \le 2\omega_i^{-1}(R)$, Thus,

$$\rho(x) = \sum_{i=1}^{n} \omega_i(|x_i|) \le \sum_{i=1}^{n} \omega_i\left(2\omega_i^{-1}(R)\right) \le 2nR.$$

On the other hand, the integral on the right-hand side of inequality (11) is actually taken over the set $E_R^{x^0}(\rho`)\backslash E_R^{x^0}(\rho)$. It follows that $x \overline{\in} E_R^{x^0}(\rho)$, i.e. $x \overline{\in} E_R^{x^0}\left(\frac{1}{2}\right)$. Then there exists $i_0, 1 \le i_0 \le n$ such that:

$$\left|x_{i_0} - x_{i_0}^0\right| \ge \frac{1}{2\sqrt{n}} \cdot \omega_{i_0}^{-1}(R).$$

At the same time, for any $i, 1 \le i \le n$

$$\left|x_i^0\right| \le \frac{1}{4\sqrt{n}} \cdot \omega_{i_0}^{-1}(R).$$

Therefore,

$$\left|x_{i_0}\right| \ge \left|x_{i_0} - x_{i_0}^0\right| - \left|x_{i_0}^0\right| \ge \frac{1}{4\sqrt{n}} \cdot \omega_{i_0}^{-1}(R)$$

and thus,

$$\rho(x) = \sum_{i=1}^{n} \omega_i(|x_i|) \ge \omega_{i_0}\left(\frac{1}{4\sqrt{n}} \cdot \omega_{i_0}^{-1}(R)\right).$$

Let l be the smallest natural number for which $4^{l-2} \ge n$, i.e. $\frac{1}{4\sqrt{n}} \ge 2^{-l}$. From condition (3), we have:

$$\omega_{i_0}\left(\frac{1}{4\sqrt{n}} \cdot \omega_{i_0}^{-1}(R)\right) \ge 2^{-1} \cdot \omega_{i_0}\left(\frac{2}{4\sqrt{n}} \cdot \omega_{i_0}^{-1}(R)\right) \ge \dots \ge$$

$$\ge 2^{-l} \cdot \omega_{i_0}\left(\frac{2^l}{4\sqrt{n}} \cdot \omega_{i_0}^{-1}(R)\right) \ge 2^{-l} \cdot R.$$

Thus, $\rho(x) \ge c_3(n) \cdot R$.

We obtain for $i = 1, \dots, n$

$$\lambda_i(x) \le \left(\frac{\omega_i^{-1}(R + 2nR)}{R + c_3 R}\right)^2 \le c_4(n) \cdot \left(\frac{\omega_i^{-1}(R)}{R}\right)^2. \qquad (12)$$

Using (10) and (11), we conclude

$$\int_{E_R^{x^0}(\rho)} \sum_{i=1}^{n} \lambda_i(x) u_i^2 \, dx \le \frac{4\gamma^{-2} \cdot c_2^2 \cdot n \cdot c_4}{(\rho` - \rho)^2 \cdot R^2} \cdot \int_{E_R^{x^0}(\rho`)} u^2 dx.$$

Lemma 4 is proven.

Set

$$\Pi_{R:\beta}(y) = \{x: |x_i - y_i| \leq \beta \cdot \omega_i^{-1}(R), i = 1, \dots, n\},$$

where $y \in E_n, \beta > 0, R \geq 1$. Let $y \in \Pi_{R:\frac{1}{4}}(0)$,

$$\Pi_{R:1}(y) = \{x: |x_i - y_i| \leq \omega_i^{-1}(R), i = 1, \dots, n\},$$

$$\Pi_R`(y) = \left\{x : \frac{1}{2}\omega_i^{-1}(R) \leq |x_i - y_i| \leq \omega_i^{-1}(R), i = 1, \dots, n\right\}.$$

It is obvious that $\Pi_R`(y) \subset \Pi_{R:1}(y)$. Let us make the substitution $z_i = \frac{x_i}{\omega_i^{-1}(R)}$, $i = 1, \dots, n$. Then $\Pi_R`(y)$ will be compact

$$\Pi_R`(y) = \left\{x : \frac{1}{2}\omega_i^{-1}(R) \leq |x_i - y_i| \leq \omega_i^{-1}(R), i = 1, \dots, n\right\}$$

Here $\tilde{y}$ is the image of the point y. Next, $x \in \Pi_R`(y)$ and $i = 1, \dots, n$ we have:

$$\lambda_i(x) = \left(\frac{\omega_i^{-1}(R + \rho(x))}{R + \rho(x)}\right)^2 = \left[\frac{\omega_i^{-1}(R + \sum_{k=1}^n \omega_k(|x_k|))}{R + \sum_{k=1}^n \omega_k(|x_k|)}\right]^2 =$$

$$= \frac{\left[\omega_i^{-1}\left(R + \sum_{k=1}^n \omega_k\left(\omega_k^{-1}(R) \cdot |z_k|\right)\right)\right]^2}{[R + \sum_{k=1}^n \omega_k(\omega_k^{-1}(R) \cdot |z_k|)]^2}, \qquad (13)$$

where $z \in \Pi_R`(\tilde{y})$. On the other hand, if $t \in (0,1)$ and the natural number l is such that $2^{-l-1} \leq t < 2^{-l}$, then for $i = 1, \dots, n$. According to condition (3)

$$\omega_i(tR) \geq \frac{1}{2}\omega_i(2tR) \geq 2^{-l-1}\omega_i\left(2^{l+1}tR\right) \geq$$

$$\geq 2^{-l-1}\omega_i(R) \geq \frac{t}{2}\omega_i(R).$$

It is easy to see that a similar estimate holds for $t = 0$ and $t = 1$. Thus,

$$R + \sum_{k=1}^n \omega_k\left(\omega_k^{-1}(R) \cdot |z_k|\right) \geq R + \frac{1}{2} \cdot \sum_{k=1}^n |z_k| \cdot \omega_k\left(\omega_k^{-1}(R)\right) =$$

$$= R + \frac{R}{2}\sum_{k=1}^n |z_k|,$$

$$\omega_i^{-1}\left(R + \sum_{k=1}^n \omega_k\left(\omega_k^{-1}(R) \cdot |z_k|\right)\right) \geq \omega_i^{-1}\left(R + \frac{R}{2} \cdot \sum_{k=1}^n |z_k|\right). \qquad (14)$$

But for $k = 1, \dots, n$

$$|z_k| \geq |z_k - \tilde{y}_k| - |\tilde{y}_k| \geq \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

Therefore, from (14), we conclude:

$$\left[\omega_i^{-1}\left(R + \sum_{k=1}^n \omega_k\left(\omega_k^{-1}(R) \cdot |z_k|\right)\right)\right]^2 \geq$$

$$\geq \left[\omega_i^{-1}\left(R+\frac{nR}{8}\right)\right]^2 \geq c_5(n)\cdot\left(\omega_i^{-1}(R)\right)^2. \qquad (15)$$

Similarly, for $k = 1, \dots, n$

$$|z_k| \geq |z_k - y_k| + |y_k| \leq 1 + \frac{1}{4} = \frac{5}{4}.$$

We obtain:

$$\left[R+\sum_{k=1}^{n}\omega_k\left(\omega_k^{-1}(R)\cdot|z_k|\right)\right]^2 \leq \left(R+\sum_{k=1}^{n}\omega_k\left(\frac{5}{4}\omega_k^{-1}(R)\right)\right)^2 \leq$$

$$\leq \left(R+2\cdot\sum_{k=1}^{n}\omega_k\left(\omega_k^{-1}(R)\right)\right)^2 = (R+2nR)^2 \leq c_6(n)\cdot R^2. \qquad (16)$$

Using (15) and (16) in (13), we arrive at the estimate:

$$\lambda_i(x) \geq c_7(n)\cdot\left(\frac{\omega_i^{-1}(R)}{R}\right)^2, \qquad (17)$$

if $x \in \Pi_R`(y)$ and $i = 1, \dots, n$.

This proves that for $x \in \Pi_R`(y), R \geq 1$ and $i = 1, \dots, n$, estimates

$$c_8(n)\cdot\left(\frac{\omega_i^{-1}(R)}{R}\right)^2 \leq \lambda_i(x) \leq c_9(n)\cdot\left(\frac{\omega_i^{-1}(R)}{R}\right)^2. \qquad (18)$$

In any limited subdomain of the domain D, equation (1) is not degenerate. Therefore, for its solutions in this subdomain, the a priori estimate of the Hölder norm and the Harnack-type inequality hold.

Theorem 1. The first boundary value problem (5), (7) is uniquely solvable in space $W_{2,\lambda}^1(D)$ for all $\varphi(x) \in W_{2,\lambda}^1(D), f(x) \in L_2(D)$.

Theorem 2. If $u(x) \in W_{2,\lambda}^1(D)$ is an L-subelliptic function that is non-positive in the sense of $W_{2,\lambda}^1(D)$ sense on ∂D, then $u(x)$ is non-positive almost everywhere in D.

Let $h(x)$ be a function continuous on the boundary of ∂D. We will seek a solution to the Dirichlet problem

$$Lu = 0, x \in D, u|_{\partial D} = h. \qquad (19)$$

If $h(x)$ is the trace on ∂D of a function $\bar{h}(x) \in W_{2,\lambda}^1(D)$, then Theorem 1 provides a solution $u(x) \in W_{2,\lambda}^1(D)$, which is locally Hölder continuous. The solution assumes boundary values in the sense of $u - \bar{h} \in \overset{\circ}{W}{}_{2,\lambda}^1(D)$. Let us denote by $\tau_\lambda^{1,2}$ the factor space $W_{2,\lambda}^1(D)\backslash\overset{\circ}{W}{}_{2,\lambda}^1(D)$, equipped with the norm

$$\|h\|_{\tau^{1,2}} = \max_{\bar{\bar{h}}-\bar{h}\in\overset{\circ}{W}{}_{2,\lambda}^1(D)} \|\bar{\bar{h}}\|_{W_{2,\lambda}^1(D)}$$

Let us denote the continuous linear mapping $\tau_\lambda^{1,2}$ in $W_{2,\lambda}^1(D)$ by the formula

$$u = Bh \tag{20}$$

By Theorem 2, if $h(x)$ is bounded on ∂D in the $W_{2,\lambda}^1(D)$ sense , then

$$\min_{\partial D} h(x) \le \min_D Bh \le \max_D Bh \le \max_{\partial D} h(x), \tag{21}$$

where $\max\limits_{\partial D} h\ \left(\min\limits_{\partial D} h(x)\right)$ denotes the maximum (minimum) of numbers ψ such that $\bar{h} \le \psi(\bar{h} \ge \psi)$ on ∂D in the sense of $W_{2,\lambda}^1(D)$.

By virtue of (21) and Lemma 4, we obtain

$$\|Bh\| \le c_{10}(\gamma, D) \cdot \max_{\partial D}|h|, \tag{22}$$

where

$$\||g|\| = \sup_{\overline{D}` \subset D} \delta \left( \sum_{i=1}^{n} \int_{D`} \lambda_i(x) g_i^2 dx \right)^{1/2} + \max_D |g|,$$

δ -is the distance between $\overline{D}`$ and ∂D.

Consequently, Bh is a linear mapping from the subset $B_\lambda^{1,2}$ of functions from $\tau_\lambda^{1,2}$ that are bounded on ∂D (in the sense of $W_{2,\lambda}^1(D)$) into the space of functions with a finite norm $\||u|\|$.

Since any continuous function $h(x)$ on ∂D can be approximated in the $\max\limits_{\partial D}|h|$ norm by functions smooth on any set containing $\overline{D}$, the set $B_\lambda^{1,2}$ is dense in the space of functions $h(x)$ continuous on ∂D with the norm $\max\limits_{\partial D}|h|$. Therefore, the linear mapping $u = Bh$, considered on the set of functions from $\tau_\lambda^{1,2}$ that are continuous on ∂D, can be extended to the entire space of functions continuous on ∂D. The mapping obtained in this way will still be denoted by

$$u = bh.$$

It associates to each function $h(x)$ continuous on ∂D a unique function $u(x)$ such that

$$\|u\| < \infty.$$

It is easy to prove that the function $u(x)$ is a local solution of the equation $Lu = 0$. Thus, we have proved the following theorem.

Theorem 3. There exists a mapping Bh that associates to each continuous function $h(x)$ on ∂D a local solution $u(x)$ of equation (1), such that if $h(x)$ is the trace of a function from $C^1(D)$, then $u = Bh$ coincides with the solution in $W_{2,\lambda}^1(D)$. Moreover, equalities (21) and (22) hold.

2. Boundary value problem and regularity

Our goal is to investigate whether $u = Bh$ assumes the boundary value $h(x)$.

Definition 7. A point $y \in \partial D$ is called regular if for any function continuous on ∂D, the equality

$$\lim_{\substack{x \to y \\ x \in D}} u(x) = h(y). \tag{23}$$

holds for the generalized solution $u = Bh$. If there exists at least one continuous function $h(x)$ on ∂D for which (23) is not satisfied, the point y is called irregular.

Lemma 5. If at a point $y \in \partial D$ equality (23) holds for $u = Bh$ for any continuous $h(x) \in \tau_\lambda^{1,2}$, then y is regular.

Proof. Let $h(x) \in C(\partial D)$. For any $\varepsilon > 0$, there exists a continuous function $h_\varepsilon(x) \in \tau_\lambda^{1,2}$ such that

$$|h(x) - h_\varepsilon(x)| < \frac{\varepsilon}{3} \quad \text{in} \quad \partial D$$

Let $u = Bh$, $u_\varepsilon = Bh_\varepsilon$. By the maximum principle (21), $|u - u_\varepsilon| < \frac{\varepsilon}{3}$ in D. From (23), it follows that there exists a neighborhood N_ε of the point y in D such that

$$|h_\varepsilon(y) - u_\varepsilon(x)| < \frac{\varepsilon}{3} \quad \text{for} \quad x \in N_\varepsilon.$$

Then

$$|h(y) - u(x)| \le |h(y) - h_\varepsilon(y)| + |h_\varepsilon(y) - u_\varepsilon(x)| + |u_\varepsilon(x) - u(x)| < \varepsilon$$

for $x \in N_\varepsilon$, so that y is regular.

Definition 8. A function $v_y(x) \in W_{2,\lambda}^1(D)$ is called a barrier at a point $y \in \partial D$, if

1) $L(v_y) = 0$ in D in the sense of the definition $\overset{\circ}{W}{}_{2,\lambda}^1(D)$;
2) for any $\rho > 0$, there exists a number $m > 0$ such that $v_y(x) > m$ in the $\overset{\circ}{W}{}_{2,\lambda}^1(D)$ sense on the set $\{x\colon\ x \in \partial D,\ |x - y| \ge \rho\}$;
3) $v_y(x)$ is continuous at the point y, and

$$\lim_{\substack{x \to y \\ x \in D}} v_y(x) = v_y(y) = 0.$$

Condition 3) makes sense, because by virtue of 1), the function $v_y(x)$ can be defined at any point of D and is continuous in D.

Lemma 6. A point $y \in \partial D$ is regular if and only if a barrier $v_y(x)$ exists at it.

Proof. If y is a regular point, then the function

$$v_y = B(|x - y|)$$

is a barrier, since the function $|x - y| \in \tau_\lambda^{1,2}$ is continuous on ∂D. Suppose now that a barrier v_y exists. Let $h(x)$ be any continuous function in $\tau_\lambda^{1,2}$ and $u = Bh$. For any $\varepsilon > 0$, there exists $\rho > 0$ such that

$$|h(x)-h(y)|<\frac{\varepsilon}{2},$$

when $|x-y|<\rho,\ x\in\partial D$. Moreover, the function $h(x)$ is bounded, so $|h(x)|\le M$ on ∂D. Since $v_y(x)\ge m$ as $|x-y|\ge\rho$, then

$$h(y)+\frac{\varepsilon}{2}+\frac{2M}{m}v_y(x)-h(x)\ge 0$$

on ∂D in the $W^1_{2,\lambda}(D)$ sense. By Theorem 2,

$$u(x)\le h(y)+\frac{\varepsilon}{2}+\frac{2M}{m}v_y(x).$$

Similarly,

$$u(x)\le h(y)-\frac{\varepsilon}{2}-\frac{2M}{m}v_y(x).$$

Since $v_y(x)\to 0$ if $x\to y$, we can find a neighborhood N_ε of the point y such that

$$2M\cdot v_y(x)<\frac{\varepsilon\cdot m}{r}\qquad \text{for}\quad x\in N_\varepsilon$$

Then

$$|u(x)-h(y)|<\varepsilon\qquad \text{for}\quad x\in N_\varepsilon$$

and (23) holds for the continuous function $h\in\tau_\lambda^{1,2}$. Therefore, by Lemma 5, the point y is regular.

Lemma 7. Let D' be a subdomain of the region D and y a common boundary point of D and D'. Suppose also that for some ball $Q(y,\rho)=\{x\colon |x-y|<\rho\}$ the equality

$$\partial D\cap Q(y,\rho)=\partial D`\cap Q(y,\rho)$$

holds. Then y is a regular boundary point of the region D if and only if it is a regular point of the subdomain D'.

Proof. First, suppose that y is regular with respect to D'. We want to show that $v_y=B(|x-y|)$ is a barrier in D.

It is clear that this function is bounded. Therefore, $v_y(x)\le c_{11}|x-y|$ on $\partial D'-(\partial D\cap\partial D`)$, where $c_{11}$ is some constant. By the maximum principle (21), $0\le v_y(x)\le c_{11}\cdot v_y`(x)$, where $v_y`(x)$ is the corresponding function $B'(|x-y|)$ in D'. Since y is regular with respect to D', then $v_y(x)\to 0$ as $x\to y$.

Consequently, $v_y(x)\to 0$ as $x\to y$ and $v_y(x)$ is a barrier.

Now suppose that y is regular with respect to D. Define the sets

$$E_\rho=(E_n\backslash D)\cap Q(y,\rho)\quad \text{and}\quad D_\rho=E_n\backslash E_\rho.$$

Let $E\subset D$ be a compact set, and let

$$U=\{\varphi(x)\colon \varphi(x)\in \overset{\circ}{W}{}^1_{2,\lambda}(D),\varphi(x)\ge 1 \text{ on } E \text{ in the sense of } \overset{\circ}{W}{}^1_{2,\lambda}(D)\}$$

We consider for $\varphi\in U$ the following functional:

$$D_L(\varphi) = \int_D \sum_{i,j=1}^{n} a_{ij}(x)\varphi_i\varphi_j\, dx.$$

The $\inf_{\varphi \in U} D_L(\varphi)$ is called the L-capacity of the compact set E relative to the domain D and is denoted by $\mathrm{cap}_L^{(D)}(E)$. The L-capacity of the compact set E relative to E_n is simply called its L-capacity and is denoted by $\mathrm{cap}_L\ (E)$.

Theorem 4. There exists a unique function $u(x) \in U$ such that $D_L(u) = \mathrm{cap}_L^{(D)}(E)$. Moreover, $u(x)\ =\ 1$ on E in the $\overset{\circ}{W}{}^1_{2,\lambda}(D)$ sense.

Proof. The proof is carried out as follows: from the convexity and closedness of the set U, the convexity of the functional $D_L(\varphi)$, and the fact that the space $\overset{\circ}{W}{}^1_{2,\lambda}(D)$ is a Hilbert space, it follows that the extreme function $u(x)$ exists and is unique. Furthermore, $u(x) \geq 1$ on E in the sense of $\overset{\circ}{W}{}^1_{2,\lambda}(D)$, then $\{u(x)\}^1 = 1$ and $\{u(x)\}^1 \in U$, $D_L(\{u\}^1) \leq D_L(u)$. From this we conclude that $D_L(\{u\}^1) = D_L(u)$ and, due to the uniqueness of the extreme function $\{u\}^1 = u$, i.e. $u(x) = 1$ on E in the sense of $\overset{\circ}{W}{}^1_{2,\lambda}(D)$.

Definition 9. The function $u(x)$ that minimizes the functional $D_L(\varphi)$ on the set U is called the capacitary potential of the compact set E.

Lemma 8. Let $u(x)$ be the capacitor potential of the compact set E. Then, if $\varphi(x) \in \overset{\circ}{W}{}^1_{2,\lambda}(D)$ and $\varphi(x) \geq 0$ on E in the sense $\overset{\circ}{W}{}^1_{2,\lambda}(D)$, then

$$\int_D \sum_{i,j=1}^{n} a_{ij}(x)u_i\varphi_j\, dx \geq 0. \tag{24}$$

Proof. Fix an arbitrary $\varepsilon > 0$ and consider the function $v(x)\ =\ u(x)\ +\ \varepsilon \cdot \varphi(x)$. It is clear that $v(x) \in \overset{\circ}{W}{}^1_{2,\lambda}(D)$ and $v(x) \geq 1$ on E in the sense $\overset{\circ}{W}{}^1_{2,\lambda}(D)$.

From this, it follows that $v(x)\ \in\ U$ and

$$\int_D \sum_{i,j=1}^{n} a_{ij}(x)\cdot(u_i + \varepsilon\varphi_i)\cdot\left(u_j + \varepsilon\varphi_j\right) dx \geq \int_D \sum_{i,j=1}^{n} a_{ij}(x)u_iu_j\, dx,$$

i.e.

$$2 \cdot \int_D \sum_{i,j=1}^{n} a_{ij}(x)u_i\varphi_j\, dx + \varepsilon \int_D \sum_{i,j=1}^{n} a_{ij}(x)\varphi_i\varphi_j\, dx.$$

Now it is sufficient to let ε tend to zero, and the inequality (24) is proven.

From Lemma 8, it follows that the capacitary potential $u(x)$ of the compact set E is an L-superelliptic function in D, equal to 1 on E in the sense $\overset{\circ}{W}{}^1_{2,\lambda}(D)$. Moreover, in (24) instead of $\varphi(x)$, one can take any function from $C^1(\overline{D})$ with compact support in $D\backslash E$. Then $u(x)$ is a generalized solution in $D\backslash E$ of the first boundary value problem

$$Lu = 0, x \in D\backslash E,$$
$$u|_{\partial E} = 1, u|_{\partial D} = 0.$$

Definition 10. Let $\mu(x)$ be a measure of bounded variation on D. A function $u(x) \in L_1(D)$ is called an L_1-solution of the equation $Lu = -\mu(x)$, equal to zero on ∂D, if for any function $\varphi(x) \in \overset{\circ}{W}{}^1_{2,\lambda}(D) \cap C(\overline{D})$ from such that $L\varphi \in C(\overline{D})$, the integral identity

$$\int_D u \cdot L\varphi dx = -\int_D \varphi d\mu$$

holds.

According to Theorem 1, there exists a linear continuous operator $G: W^{-1}_{2,\lambda}(D) \to \overset{\circ}{W}{}^1_{2,\lambda}(D)$ such that for any functional $T \in W^{-1}_{2,\lambda}(D)$ the function $u = G(T)$ is the unique generalized solution from $\overset{\circ}{W}{}^1_{2,\lambda}(D)$ of the equation $Lu = T$. The operator G is called the Green's operator.

It is easy to see that $u(x)$ is an L_1-solution to the equation $Lu = -\mu$ in D, vanishing at ∂D, if and only if, for any function $\psi(x) \in C(\overline{D})$, the following identity

$$\int_D u \cdot \psi dx = -\int_D G(\psi) d\mu \qquad (25)$$

holds.

Remark. Let $u(x) \in \overset{\circ}{W}{}^1_{2,\lambda}(D)$ and for any function $\varphi(x) \in \overset{\circ}{W}{}^1_{2,\lambda}(D) \cap C(\overline{D})$ with $L\varphi \in C(\overline{D})$, the integral identity

$$\int_D \sum_{i,j=1}^{n} a_{ij}(x) u_i \varphi_j \, dx = \int_D \varphi \, d\mu \qquad (26)$$

holds.

Then the function $u(x)$ is an L_1-solution of the equation $Lu = -\mu$, equal to zero on ∂D.

Indeed, let $\varphi = G(\psi)$, where $\psi \in C(\overline{D})$. According to (26), we have

$$\int_D u\psi\, dx = \int_D u \cdot L\varphi\, dx = \int_D u \cdot \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial \varphi}{\partial x_j}\right) dx =$$

$$= -\int_D \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_j}\, dx =$$

$$= -\int_D \varphi\, d\mu = -\int_D G(\psi)\, d\mu, \qquad (27)$$

and the required assertion follows from (25).

On the other hand, from (27) it follows that

$$\left|\int_D \varphi\, d\mu\right| \le \int_D \left|\sum_{i,j=1}^{n} a_{ij}(x) u_i \varphi_j\right| dx \le$$

$$\le \left(\int_D \sum_{i,j=1}^{n} a_{ij}(x) u_i u_j\, dx\right)^{1/2} \cdot \left(\int_D \sum_{i,j=1}^{n} a_{ij}(x) \varphi_i \varphi_j\, dx\right)^{1/2} \le$$

$$\le \gamma^{-1} \cdot \left(\int_D \sum_{i=1}^{n} \lambda_i(x) u_i^2\, dx\right)^{1/2} \cdot \left(\int_D \sum_{i,j=1}^{n} \lambda_i(x) \varphi_i^2\, dx\right)^{1/2} \le$$

$$\le \gamma^{-1} \cdot \|u\|_{W_{2,\lambda}^1(D)} \cdot \|\varphi\|_{W_{2,\lambda}^1(D)} = c_{12}(\gamma, n) \cdot \|\varphi\|_{W_{2,\lambda}^1(D)}.$$

The mapping $\mu \to u$ determines the operator $G*$, the adjoint of G, i.e., $u = G^*(\mu)$.

Since $G\left(W_{p,\lambda}^{-1}(D)\right) \subset C(\overline{D})$ and G^* are defined on the space adjoint to $G\left(W_{p,\lambda}^{-1}(D)\right)$, it is understood that G^* is defined for all measures μ of bounded variation, and the range of the operator G^* lies in the dual of $W_{p,\lambda}^{-1}(D)$.

Thus $\mu \in W_{2,\lambda}^{-1}(D)$. This proves the following:

Theorem 5. An L_1 solution $u(x)$ of the equation $Lu = -\mu$ in D, vanishing on ∂D, belongs to the space $\overset{\circ}{W}{}_{2,\lambda}^1(D)$ if and only if $\mu \in W_{2,\lambda}^{-1}(D)$.

We will now show that a weak solution can be approximated by weak solutions of equations with continuous coefficients.

Let us take a family of functions $d_s(x)$ possessing the properties:

1) $d_s(x) \in C^\infty(E_n)$,
2) $d_s(x) \ge 0$,
3) $d_s(x) = 0$ for $|x| > \frac{1}{3}$,
4) $\int_{E_n} d_s(x) dx = 1$.

We set

$$L^{(s)}u = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}^{(s)}(x)\frac{\partial u}{\partial x_j}\right)$$

where

$$a_{ij}^{(s)}(x) = \int_{E_n} a_{ij}(y)\cdot \alpha_s(x-y)\,dy.$$

We consider the coefficients of the operator L to be extended into $E_n\backslash D$ as follows:

$$a_{ij}(x) = \delta_{ij}\lambda_i(x), \qquad i,j = 1,\dots,n.$$

Clearly, the operators $L^{(s)}$ are uniformly elliptic, since $a_{ij}^{(s)}(x) \in C^\infty(E_n)$. Furthermore, let $u_s(x)$ be the Friedrichs mollification of the function $u(x)$ with parameter $s > 0$.

Theorem 6. Let μ be a measure of bounded variation on D, and let $u_s(x)$ be an L_1 solution of the equation $L^{(s)}u_s = -\mu$ in D, vanishing at ∂D. Then there exist $p_1(\gamma,n) > 1$ and $q_1(\gamma,n) > 1$ such that the sequence $\{u_s(x)\}$ converges to the L_1-solution $u(x)$ of the equation $Lu = -\mu$ in D, vanishing on ∂D, weakly in $\overset{\circ}{W}{}^1_{p,\lambda}(D)$ and strongly in $L_q(D)$, provided $1 \le p < p_1,\ 1 \le q < q_1$.

Definition 11. Let y be an arbitrary fixed point in D. The function $g(x,y)$, which is an L_1-solution of the equation $Lg = -\delta(x-y)$ in D, vanishing on ∂D, is called the Green's function of the operator L in D.

Here $\delta(x)$ is the Dirac generalized function.

From this definition, it follows that if $\varphi(x) \in \overset{\circ}{W}{}^1_{2,\lambda}(D)$ is a generalized solution of the equation $L\varphi = -\psi$ where $\psi \in C(\overline{D})$, then it can be represented in the form

$$\varphi(y) = \int_D g(x,y)\varphi(x)\,dx. \qquad (28)$$

Theorem 7. For every measure $\mu(x)$ of bounded variation on D, the function

$$u(x) = \int_D g(x,y)\,d\mu(y) \qquad (29)$$

is finite almost everywhere in D and is an L_1-solution of the equation $Lu = -\mu$ in D, vanishing on ∂D.

Proof. Let $\psi(x) \geq 0, \psi(x) \in C(\overline{D})$, and let $\varphi(x) \in \overset{\circ}{W}{}^1_{2,\lambda}(D)$ be a generalized solution of the equation $L\varphi = -\psi$ in D. Then $\varphi(x) \in C(D)$, since ∂D is a smooth surface and the boundary function is zero. By the maximum principle, $\varphi(x) \geq 0$ in D. Moreover, the representation (28) holds for it. Thus, the integral

$$I = \int_D \varphi(y) d\mu(y) = \int_D d\mu(y) \int_D g(x,y)\psi(x)dx$$

exists. By Fubini's theorem, taking into account (28), we obtain

$$I = \int_D \psi(x)dx \int_D g(x,y)d\mu(y) = \int_D u(x)\psi(x)dx.$$

Hence

$$\int_D u \cdot \psi dx = \int_D \varphi d\mu = \int_D G(\psi) d\mu,$$

i.e., according to (25), the function $u(x)$ is an L_1-solution of the equation $Lu = -\mu$ in D, vanishing on ∂D.

Theorem 7 is proven.

Theorem 8. Let $\mu(x)$ and $\nu(x)$ be measures of bounded variation on D, and let the functions $u(x)$ and $v(x)$ be L_1-solutions of the equations $Lu = -\mu$ and $Lv = -\nu$ in D, respectively, vanishing on ∂D. Then

$$\int_D u d\nu = \int_D v d\mu = \iint_{D\times D} g(x,y) d\mu(x) d\nu(y).$$

This statement follows from Fubini's theorem.

Now let $u(x)$ be the capacitary potential of a compact set $E \subset D$. Then, by Lemma 8, for any function $\varphi(x) \in C_0^\infty(D), \varphi(x) \geq 0$ on D, the inequality

$$\int_D \sum_{i,j=1}^n a_{ij}(x) u_i \varphi_j dx \geq 0$$

holds.

Consider the functional

$$J(\varphi) = \int_D \sum_{i,j=1}^n a_{ij}(x) u_i \varphi_j dx.$$

By the Riesz representation theorem (Schwarz's theorem), there exists a measure $\mu(x)$ of bounded variation on D (with support in E) such that

$$J(\varphi)=\int_D\sum_{i,j=1}^n a_{ij}(x)u_i\varphi_j dx=\int_D\varphi d\mu. \tag{30}$$

But $u(x)=1$ on E in the $\overset{\circ}{W}{}^1_{2,\lambda}(D)$ sense. Therefore, the support of the measure $\mu(x)$ is located on ∂E.

The measure $\mu(x)$ is called the capacitary distribution of the compact set E. It is clear that the capacitary potential $u(x)$ of the compact set E is an L_1-solution of the equation $Lu=-\mu$ in D, vanishing on ∂D. Therefore

$$u(x)=\int_D g(x,y)d\mu(y).$$

On the other hand, since $u(x)=1$ on E in the $\overset{\circ}{W}{}^1_{2,\lambda}(D)$ sense, there exists a sequence $\{\varphi_m(x)\}$ such that $\varphi_m(x)\in M_0(D), \varphi_m(x)=1$ on E and

$$\lim_{m\to\infty}\|\varphi_m-u\|_{W^1_{2,\lambda}(D)}=0.$$

Let us write equality (30), substituting the function $\varphi(x)$ for $\varphi_m(x)$. Taking into account the inclusion $\operatorname{supp}\mu(x)\subset E$, we have

$$\int_D\sum_{i,j=1}^n a_{ij}(x)u_i(\varphi_m)_j dx=\int_D\varphi_m d\mu=$$

$$=\int_D\varphi_m d\mu=\int_E\varphi_m dm=\int_E d\mu=\mu(E). \tag{31}$$

Passing to the limit as $m\to\infty$ and using that

$$\int_D\sum_{i,j=1}^n a_{ij}(x)u_iu_j dx=\operatorname{cap}_L^{(D)}(E),$$

we arrive at the equality

$$\operatorname{cap}_L^{(D)}(E)=\mu(E). \tag{32}$$

It is known that for an operator L with coefficients from $C^\infty(D)$, the Green's function is non-negative and $g(x,y)=g(y,x)$. Using limiting arguments, these facts are easily extended to any non-uniformly elliptic operator L.

Since, according to Theorem 6, the Green's function of any non-uniformly elliptic operator L can be approximated, we will first consider only the operator L with coefficients from $C^\infty(\overline{D})$. Then, as is known, for $x\neq y$ the function $g(x,y)$ is continuous in x and y and

$$\lim_{x\to y}g(x,y)=+\infty.$$

Consequently, y is an interior point of the set $J_a = \{x: g(x,y) \geq a\}$, where a for some positive number a.

Consider the capacitary potential $u(x)$ of this compact set J_a. It is clear that

$$u(x) = \int_D g(z,x) d\nu_a(x),$$

where $\nu_a(x)$ is the capacitary distribution of J_a. Moreover, the function $u(x)$ is continuous at interior points of J_a, i.e. $\operatorname{supp}\nu_a(x) \subset \partial J_a$. Thus

$$1 = \int_{\partial J_a} g(z,y) d\nu_a(z).$$

On the other hand, for $z \subset \partial J_a, g(z,y) = a$. Therefore, taking into account (32), we obtain

$$\operatorname{cap}_L^{(D)}(J_a) = \frac{1}{a}. \tag{33}$$

Let $y \in \Pi_{R:\frac{1}{4}}(0)$, and we assume R is large enough that $\overline{\Pi}_{R:2}(0) \subset D$ and

$$a = \inf_{x \in \partial \Pi_{R:1}(y)} g(x,y).$$

From the maximum principle, it follows that $\Pi_{R:1}(y) \subset J_a$, and from (33), we conclude

$$\operatorname{cap}_L^{(D)}\big(\Pi_{R:1}(y)\big) \leq \operatorname{cap}_L^{(D)}(J_a) = \frac{1}{a} = \frac{1}{\inf\limits_{x \in \partial \Pi_{R:1}(y)} g(x,y)}. \tag{34}$$

Similarly, if

$$b = \sup_{x \in \partial \Pi_{R:1}(y)} g(x,t),$$

then $\Pi_{R:1}(y) \supset J_b$, where $J_b = \{x : g(x,y) \leq b\}$. Therefore

$$\operatorname{cap}_L^{(D)}\big(\Pi_{R:1}(y)\big) \leq \operatorname{cap}_L^{(D)}(J_b) = \frac{1}{b} = \frac{1}{\sup\limits_{x \in \partial \Pi_{R:1}(y)} g(x,y)}. \tag{35}$$

From (34)-(35), it follows that

$$\inf_{x \in \partial \Pi_{R:1}(y)} g(x,y) \leq \left(\operatorname{cap}_L^{(D)}\big(\Pi_{R:1}(y)\big)\right)^{-1} \leq \sup_{x \in \partial \Pi_{R:1}(y)} g(x,y). \tag{36}$$

But according to the Harnack-type inequality,

$$\sup_{x \in \partial \Pi_{R:1}(y)} g(x,y) \leq c_{13}(\gamma, n) \cdot \inf_{x \in \partial \Pi_{R:1}(y)} g(x,y). \tag{37}$$

Now, from (36)-(37) for $x \in \partial \Pi_{R:1}(y)$, we obtain

$$c_{13}^{-1} \left(\operatorname{cap}_L^{(D)}\big(\Pi_{R:1}(y)\big)\right)^{-1} \leq g(x,y) \leq c_{13} \cdot \left(\operatorname{cap}_L^{(D)}\big(\Pi_{R:1}(y)\big)\right)^{-1}. \tag{38}$$

Thus, the following is proven:

Theorem 9. If conditions (2)-(4) are satisfied for the coefficients of the operator L, and $y \in \Pi_{R:\frac{1}{4}}(y), x \in \partial\Pi_{R:1}(y)$ then for sufficiently large R, the estimates (38) are valid for the function $g(x,y)$.

Let us denote by L_0 the model operator

$$L_0 = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\lambda_i(x)\frac{\partial}{\partial x_i}\right).$$

According to (2), for any compact set $E \subset D$,

$$\gamma \cdot \text{cap}_{L_0}^{(D)}(E) \le \text{cap}_{L}^{(D)}(E) \le \gamma^{-1} \cdot \text{cap}_{L_0}^{(D)}(E). \qquad (39)$$

Setting $D = E_n$ and denoting the function $g(x,y)$ by $G(x,y)$ (in this case $G(x,y)$ is the fundamental solution of equation (1)), from (38)-(39) we introduce the following:

Corollary 1. If the conditions of Theorem 9 are satisfied, there exists a constant $c_{14}(\gamma, n)$ such that for sufficiently large R

$$c_{14}^{-1}\left(\text{cap}_{L_0}\left(\Pi_{R:1}(y)\right)\right)^{-1} \le G(x,y) \le c_{14} \cdot \left(\text{cap}_{L_0}\left(\Pi_{R:1}(y)\right)\right)^{-1}, \qquad (40)$$

where

$$\text{cap}_{L_0}(E) = \text{cap}_{L_0}^{(E_n)}(E).$$

Lemma 9. Let $y \in \Pi_{R:\frac{1}{4}}(0)$. Then for sufficiently large R, the estimate

$$\text{cap}_{L_0}\left(\Pi_{R:1}(y)\right) \le c_{15}(\gamma, n) \cdot R^{-2} \cdot \prod_{i=1}^{n} \omega_i^{-1}(R) \qquad (41)$$

holds.

Proof. Introduce the function $f^i(t), i = 1, \dots, n$ as follows: $f^i(t) = 1$, if $|t - y_i| \le \omega_i^{-1}(R); f^i(t) = 0$, if $|t - y_i| \ge 2\omega_i^{-1}(t); 0 \le f^i(t) \le 1$, $f^i(t) \in C^\infty(E_1)$ and

$$\left|\frac{df^i}{dt}\right| \le \frac{c_{16}}{\omega_i^{-1}(R)}.$$

Let it be

$$u(x) = \prod_{i=1}^{n} f^i(x_i).$$

It is clear that $u(x) = 1$ for $x \in \Pi_{R:1}(y), u(x) = 0$ for $x \overline{\in} \Pi_{R:2}(y)$, $u(x) \in \overset{\circ}{W}{}_{2,\lambda}^{1}(E_n)$ and

$$|u_i| \le \frac{c_{16}}{\omega_i^{-1}(R)}, i = 1, \dots, n. \qquad (42)$$

Taking into account (18) and (42), we have

$$\operatorname{cap}_{L_0}\big(\Pi_{R:1}(y)\big) \le D_{L_0}(u) = \int\limits_{\Pi_{R:2}(y)} \sum_{i=1}^{n} \lambda_i(x) u_i^2 dx =$$

$$= \int\limits_{\Pi_{R:2}(y)\setminus\Pi_{R:1}(y)} \sum_{i=1}^{n} \lambda_i(x) u_i^2 dx \le$$

$$\le c_9 \cdot R^{-2} \cdot c_{16}^2 \cdot n \cdot \operatorname{mes}\big(\Pi_{R:2}(y)\big) =$$

$$= 2^n \cdot c_9 \cdot c_{16}^2 \cdot n \cdot R^{-2} \cdot \prod_{i=1}^{n} \omega_i^{-1}(R).$$

Thus, the required estimate (41) is proven.

Lemma 10. Let $y \in \Pi_{R:\frac{1}{4}}(y)$. Then for sufficiently large R, the estimate

$$\operatorname{cap}_{L_0}\big(\Pi_{R:1}(y)\big) \ge c_{17}(\gamma, n) \cdot R^{-2} \cdot \prod_{i=1}^{n} \omega_i^{-1}(R) \qquad (43)$$

holds.

Proof. It is clear that $\Pi_R`(y) \subset \Pi_{R:1}(y)$ and

$$\operatorname{cap}_{L_0}\big(\Pi_{R:1}(y)\big) \ge \operatorname{cap}_{L_0}\big(\Pi_R`(y)\big) \qquad (43)$$

will replace variables $z_i = \frac{x_i}{\omega_i^{-1}(R)}, i = 1, \dots, n$. Then, in this way, $\Pi_R`(y)$ will be compact

$$\widetilde{\Pi}`(\tilde{y}) = \left\{z : \frac{1}{2} \le |z_i - \tilde{y}_i| \le 1, i = 1, \dots, n\right\}.$$

Here, $\tilde{y}$ represents point y.

Now let $u(x) \in \overset{\circ}{W}{}^1_{2,\lambda}(D)$ be an arbitrary function such that $u(x) = 1$ on $\Pi_R`(y)$ in the $\overset{\circ}{W}{}^1_{2,\lambda}(D)$ sense, and $\tilde{u}(z)$ is its image under the coordinate transformation we made. Taking into account (18), we have

$$\int\limits_{E_n} \sum_{i=1}^{n} \lambda_i(x) u_i^2 dx \ge c_8 \cdot R^{-2} \cdot \int\limits_{E_n} \sum_{i=1}^{n} \big(\omega_i^{-1}(R) \cdot u_i\big)^2 dx =$$

$$= c_8 \cdot R^{-2} \cdot \int\limits_{E_n} \sum_{i=1}^{n} \big(\omega_i^{-1}(R)\big)^2 \cdot \left(\frac{\partial \tilde{u}}{\partial z_i} \cdot \frac{1}{\omega_i^{-1}(R)}\right)^2 \cdot \prod_{i=1}^{n} \omega_i^{-1}(R)\, dz =$$

$$\le c_8 \cdot R^{-2} \cdot \prod_{i=1}^{n} \omega_i^{-1}(R) \cdot \int (\Delta \tilde{u})^2\, dz.$$

From (45), it follows that

$$\operatorname{cap}_{L_0}\big(\Pi_{R:1}(y)\big) \ge \operatorname{cap}_{L_0}\big(\Pi_R`(y)\big) \ge$$

$$\geq c_8 \cdot R^{-2} \cdot P\left(\widetilde{\Pi}`(y)\right) \cdot \prod_{i=1}^{n} \omega_i^{-1}(R),$$

where $P(E)$ is the wiener capacity of the compact set E. But

$$P\left(\widetilde{\Pi}`(y)\right) = P\left(\left\{z: \frac{1}{2} \leq z_i \leq 1, i = 1, \dots, n\right\}\right) = c_{18}(n)$$

and thus the estimate (43) is proven.

Let's now

$$\rho(x,y) = \sum_{i=1}^{n} \omega_i(|x_i - y_i|).$$

Theorem 10. If conditions (2)-(4) are satisfied for the coefficients of operator L, $x, y \in Q_\delta(0), x \neq y$, then for sufficiently small δ, the following estimates are valid for the fundamental solution $G(x,y)$

$$c_{19}^{-1}(\gamma,n) \cdot \left(\rho(x,y)\right)^2 \cdot \left(\prod_{i=1}^{n} \omega_i^{-1}\left(\rho(x,y)\right)\right)^{-1} \leq G(x,y) \leq$$

$$\leq c_{19} \cdot \left(\rho(x,y)\right)^2 \cdot \left(\prod_{i=1}^{n} \omega_i^{-1}\left(\rho(x,y)\right)\right)^{-1}. \tag{46}$$

Proof. Let x, y be located in the ball $Q\delta(0)$ and $x \neq y$. Set

$$R = \max_{1 \leq i \leq n} \omega_i(|x_i - y_i|).$$

Then there exists $i_0, 1 \leq i_0 \leq n$ such that $R = \omega_{i_0}\left(\left|x_{i_0} - y_{i_0}\right|\right)$. Thus

$$R \leq \sum_{i=1}^{n} \omega_i(|x_i - y_i|) = \rho(x,y). \tag{47}$$

On the other hand, for 1, then

$$\rho(x,y) = \sum_{i=1}^{n} \omega_i(|x_i - y_i|) \leq nR, i.e.\ R \geq \frac{\rho(x,y)}{n}. \tag{48}$$

According to Corollary 1 and Lemmas 9-10, for $x \in \partial\Pi_{R:1}(y)$ the estimates

$$c_{20}^{-1}(\gamma,n) \cdot \left[R^2 \cdot \prod_{i=1}^{n} \omega_i^{-1}(R)\right]^{-1} \leq G(x,y) \leq$$

$$\leq c_{20} \cdot \left[R^{-2} \cdot \prod_{i=1}^{n} \omega_i^{-1}(R)\right]^{-1}. \tag{49}$$

hold. Taking into account (47) and (48) in (49), we arrive at the required estimates (46).

Theorem 10 is proven.

Theorem 11. Let G be a region in $\Pi_{R:2}(0)$ with boundary points on $\partial\Pi_{R:2}(0)$ and intersecting $\Pi_{R:1}(0)$. Furthermore, let $u(x)$ be a weak solution of equation (1) defined in G, which vanishes on $\partial G \cap \Pi_{R:2}(0)$ in the sense of $\overset{\circ}{W}{}^{1}_{2,\lambda}(D)$, and let conditions (2)-(4) hold for the coefficients of the operator L. Then, if $R \geq 1$ and $H_R = \overline{\Pi}_{R:1}(0)\backslash G$, then

Proof. Let $U(x)$ be the capacitary potential of H_R with respect to $\Pi_{R:2}(0)$, $M = \sup_G u(x)$ and

$$U(x) = \int\limits_{H_R} g(x,y)d\mu(y),$$

where $g(x,y)$ is the Green's function of equation (1) in $\Pi_{R:2}(0)$ and $\mu(x)$ is the capacitary distribution of the compact H_R with respect to $\Pi_{R:2}(0)$. From what was proven earlier,

$$\mu(H_R) = cap_L^{(\Pi_{R:2}(0))}(H_R). \tag{51}$$

Consider the auxiliary function

$$W(x) = M(1 - U(x)) - u(x).$$

It is clear that $W(x)$ is a weak solution of equation (1) in G. On the part of ∂G that lies on $\partial\Pi_{R:2}(0)$, $g(x,y) = 0$ in the $\overset{\circ}{W}{}^{1}_{2,\lambda}(D)$ sense, i.e.,

$$W(x)|_{\partial G\cap\Pi_{R:2}(0)} = M\left(1 - U|_{\partial G\cap\Pi_{R:2}(0)}\right) \geq 0$$

and

$$W|_{\Gamma} = M(1 - U|_{\Gamma}) \geq 0.$$

By the maximum principle, $W(x) \geq 0$ in G, i.e., for any $x \in G$,

$$u(x) \leq M(1 - U(x))$$

and, in particular,

$$\sup_{G\cap\Pi_{R:1}(0)} u(x) \leq \sup_{G\cap\Pi_{R:2}(0)} u(x) \leq$$

$$\leq M\left(1 - \inf_{\substack{x,y\in\Pi_{R:1}(0)\\ x\neq y}} g(x,y)\cdot cap_L^{(\Pi_{R:2}(0))}(H_R)\right). \tag{52}$$

On the other hand,

$$\inf_{\substack{x,y\in\Pi_{R:1}(0)\\ x\neq y}} g(x,y) \geq$$

$$\geq c_{22}(\gamma,n)\cdot(\rho(x,y))^2\cdot\left(\prod_{i=1}^{n}\omega_i^{-1}(\rho(x,y))\right)^{-1}. \tag{53}$$

Furthermore, we have $x,y \in \Pi_{R:1}(0)$ and $i = 1,\ldots,n$

$$|x_i - y_i| \le |x_i| + |y_i| \le 2\omega_i\ (R),$$
$$\rho(x,y) = \sum_{i=1}^{n} \omega_i\ (|x_i - y_i|) \le \sum_{i=1}^{n} \omega_i^{-1}\left(2\omega_i^{-1}R\right) \le 2n \cdot R.$$

From (53), we conclude

$$\inf_{\substack{x,y\in\Pi_{R:1}(0)\\ x\neq y}} g(x,y) \ge$$
$$\ge c_{23}(\gamma,n)\cdot R^2\cdot\left(\prod_{i=1}^{n}\omega_i^{-1}(R)\right)^{-1}. \tag{54}$$

Substituting (54) into (52), we obtain

$$\sup_{G\cap\Pi_{R:1}(0)} u(x) \le M \times$$
$$\times\left(1 - c_{23}\cdot R^2\cdot\left(\prod_{i=1}^{n}\omega_i^{-1}(R)\right)^{-1}\cdot cap_L^{(\Pi_{R:2}(0))}(H_R)\right).$$

This completes the proof of Theorem 11.

Corollary 2. If the conditions of Theorem 11 are satisfied, then

$$\sup_G u(x) \ge \left[1 + c_{24}(\gamma,n)\cdot R^2\cdot\left(\prod_{i=1}^{n}\omega_i^{-1}(R)\right)^{-1}\cdot cap_{L_0}(H_R)\right]\times$$
$$\times \sup_{G\cap\Pi_{R:1}(0)} u(x). \tag{55}$$

3. Phragmén–Lindelöf type theorems

Theorem 12. Let $u(x) \in W_{2,\lambda}^{1,loc}(D)$ be a weak solution of equation (1) in $D, u|_{\partial D} \le 0$ in the sense of $W_{2,\lambda}^{1}(D)$, and let conditions (2)-(4) hold for the coefficients of the operator L. Then

1) either $u(x) \le 0$ in D,
2) either

$$\varliminf_{R\to\infty} M(R)/\exp\left[c(\gamma,n)\cdot\sum_{m=1}^{[\ln R]} A^{2m}\times\right.$$
$$\left.\times\left(\prod_{i=1}^{n}\omega_i^{-1}(A^m)\right)^{-1}\cdot cap_L^{\left(\Pi_{A^m:2}(0)\right)}(H^m)\right] > 0,$$

where

$$M(R) = \sup_{D\cap\Pi_{R:1}(0)} u(x), A \ge 2, H^m = \overline{\Pi}_{A^m:1}(0)\backslash D, m = 1,2, \ldots$$

Proof. Suppose alternative 1) does not hold, then there exists a point $x^0 \in D$ such that $u(x^0) > 0$. Let m_1 be the smallest natural number for which $x^0 \in \Pi_{A^m:1}(0)$ and let

$$M_i = \sup_{D \cap \Pi_{A^i:1}(0)} u(x) \quad \text{for} \quad i = 1,2,\dots$$

According to Theorem 11,

$$M_{i+1} \geq \left[1 + c_{21} \cdot A^{2i} \cdot \left(\prod_{j=1}^{n} \omega_j^{-1}(A^i) \right)^{-1} \cdot cap_L^{\left(\Pi_{A^i:2}(0)\right)}(H^i) \right] M_i \quad (56)$$

Let $R \geq 1$ sufficiently large, be $m > m_1$ a natural number for which

$$\overline{\Pi}_{A^m:1}(0) \subset \overline{\Pi}_{R:1}(0) \subset \overline{\Pi}_{A^{m+1}:1}(0),$$

i.e.

$$\omega_i^{-1}(A^m) \leq \omega_i^{-1}(R) \leq \omega_i^{-1}(A^{m+1})$$

and

$$A^m \leq R \leq A^{m+1}. \quad (57)$$

Applying inequality (56) successively, we obtain

$$M(R) \geq$$

$$\geq \prod_{i=m_1}^{m-1} \left[1 + c_{21} \cdot A^{2i} \cdot \left(\prod_{j=1}^{n} \omega_j^{-1}(A^i) \right)^{-1} \cdot cap_L^{\left(\Pi_{A^i:2}(0)\right)}(H^i) \right] \cdot u(x^0),$$

therefore,

$$\geq \sum_{i=m_1}^{m-1} \ln \left(1 + c_{21} \cdot A^{2i} \cdot \left(\prod_{j=1}^{n} \omega_j^{-1}(A^i) \right)^{-1} \cdot cap_L^{\left(\Pi_{A^i:2}(0)\right)}(H^i) \right) +$$

$$+ \ln u(x^0),$$

i.e.,

$$M(R) \geq u(x^0) \times$$

$$\times \exp \left[\sum_{i=m_1}^{m-1} \ln \left(1 + c_{21} \cdot A^{2i} \cdot \left(\prod_{j=1}^{n} \omega_j^{-1}(A^i) \right)^{-1} \times \right.\right.$$

$$\left.\left. \times\, cap_L^{\left(\Pi_{A^i:2}(0)\right)}(H^i) \right) \right]. \quad (58)$$

It is known that

$$cap_L^{\left(\Pi_{A^i:2}(0)\right)}\left(H^i\right) \leq cap_L^{(\ \ )}\left(\Pi_{A^i:1}(0)\right) \leq$$
$$\leq c_{21}(\gamma, n) \cdot A^{2i} \cdot \prod_{j=1}^{n} \omega_j^{-1}\left(A^i\right). \tag{59}$$

On the other hand, for $t \in [0, c_{25}]$

$$\ln(1 + c_{21} \cdot t) \geq c_{26}(c_{25}, c_{21}) \cdot t.$$

Therefore, from (58), it follows that

$$M(R) \geq a \cdot \exp\left[c_{26} \cdot \sum_{i=m_1}^{m-1} A^{2i} \cdot \left(\prod_{j=1}^{n} \omega_j^{-1}\left(A^i\right)\right)^{-1} \cdot cap_L^{\left(\Pi_{A^i:2}(0)\right)}\left(H^i\right)\right] =$$
$$= a \cdot \exp\left[\frac{c_{26}}{2} \cdot \sum_{i=1}^{m-1} A^{2i} \cdot \left(\prod_{j=1}^{n} \omega_j^{-1}\left(A^i\right)\right)^{-1} \cdot cap_L^{\left(\Pi_{A^i:2}(0)\right)}\left(H^i\right)\right] \times$$
$$\times \exp\left[\frac{c_{26}}{2} \cdot \left(\sum_{i=1}^{m-1} A^{2i} \cdot \left(\prod_{j=1}^{n} \omega_j^{-1}\left(A^i\right)\right)^{-1} \cdot cap_L^{\left(\Pi_{A^i:2}(0)\right)}\left(H^i\right) -\right.\right.$$
$$\left.\left. - 2 \cdot \sum_{i=1}^{m-1} A^{2i} \cdot \left(\prod_{j=1}^{n} \omega_j^{-1}\left(A^i\right)\right)^{-1} \cdot cap_L^{\left(\Pi_{A^i:2}(0)\right)}\left(H^i\right)\right)\right], \tag{60}$$

where $a = u(x^0)$.

It is easy to see that alternative 2) only makes sense if

$$\sum_{i=1}^{\infty} A^{2i} \cdot \left(\prod_{j=1}^{n} \omega_j^{-1}\left(A^i\right)\right)^{-1} \cdot cap_L^{\left(\Pi_{A^i:2}(0)\right)}\left(H^i\right) = \infty. \tag{61}$$

Therefore, considering condition (61) to be satisfied, we conclude that for sufficiently large m (i.e., R)

$$\sum_{i=1}^{m-1} A^{2i} \cdot \left(\prod_{j=1}^{n} \omega_j^{-1}\left(A^i\right)\right)^{-1} \cdot cap_L^{\left(\Pi_{A^i:2}(0)\right)}\left(H^i\right) \geq$$
$$\geq 2 \cdot \sum_{i=1}^{m-1} A^{2i} \cdot \left(\prod_{j=1}^{n} \omega_j^{-1}\left(A^i\right)\right)^{-1} \cdot cap_L^{\left(\Pi_{A^i:2}(0)\right)}\left(H^i\right).$$

Thus, from (60), we obtain

$$M(R) \geq$$

$$\geq a \cdot \exp\left[\frac{c_{26}}{2} \cdot \sum_{i=1}^{m-1} A^{2i} \cdot \left(\prod_{j=1}^{n} \omega_j^{-1}(A^i)\right)^{-1} \cdot cap_L^{\left(\Pi_{A^i:2}(0)\right)}(H^i)\right]. (62)$$

Now, taking into account that according to (57), $m-1 \geq [\ln R]-2$. From (62) and (59), we have

$$M(R) \geq a \times$$

$$\times \exp\left[\frac{c_{26}}{2} \cdot \sum_{i=1}^{[\ln R]-2} A^{2i} \cdot \left(\prod_{j=1}^{n} \omega_j^{-1}(A^i)\right)^{-1} \cdot cap_L^{\left(\Pi_{A^i:2}(0)\right)}(H^i)\right] =$$

$$= a \cdot \exp\left[\frac{c_{26}}{2} \cdot \sum_{i=1}^{[\ln R]} A^{2i} \cdot \left(\prod_{j=1}^{n} \omega_j^{-1}(A^i)\right)^{-1} \cdot cap_L^{\left(\Pi_{A^i:2}(0)\right)}(H^i)\right] \times$$

$$\times \exp\left[-\frac{c_{26}}{2} \cdot \sum_{i=[\ln R]-1}^{[\ln R]} A^{2i} \cdot \left(\prod_{j=1}^{n} \omega_j^{-1}(A^i)\right)^{-1} \cdot cap_L^{\left(\Pi_{A^i:2}(0)\right)}(H^i)\right] \geq$$

$$\geq a \cdot \exp(-c_{25}, c_{26}) \times$$

$$\times \exp\left[\frac{c_{26}}{2} \cdot \sum_{i=1}^{[\ln R]} A^{2i} \cdot \left(\prod_{j=1}^{n} \omega_j^{-1}(A^i)\right)^{-1} \cdot cap_L^{\left(\Pi_{A^i:2}(0)\right)}(H^i)\right].$$

Thus, alternative 2) holds with $c(\gamma, n) = \frac{c_{26}}{2}$.

Theorem 12 is proven.

Theorem 13. Let $h > 1$, K_h-double cone $\{x: x_n^2 \geq h^2 \cdot \sum_{i=1}^{n-1} x_i^2\}$, $D = E_n \backslash K_n$. Then, if $u(x) \in W_{2,\lambda}^{1,loc}(D)$ is a weak solution of equation (1) in D, $u|_{\partial D} \leq 0$ in the sense of $W_{2,\lambda}^1(D)$, and conditions (2)-(4) are satisfied with respect to the coefficients of the operator L, then

1) either $u(x) \leq 0$ in D,
2) either

$$\varliminf_{R\to\infty} M(R)/\exp\left[\eta_1 \cdot [\ln R]\right] > 0,$$

where $\eta_1 = \eta_1(\gamma, n)$.

Proof. Let $R \geq 1, h > 1, x^0 \in D$,

$$\Pi_{\rho:1}(x^0) = \left\{x: \left|x_i - x_i^0\right| \leq \omega_i^{-1}(\rho), i = 1, \dots, n, \rho \geq 1\right\},$$

$$\Pi_{R:1}(x^0) = \left\{x: \left|x_i - x_i^0\right| \leq \omega_i^{-1}(\rho), i = 1, \dots, n\right\}$$

and $\Pi_{\frac{R}{4h}:1}(x^0) \subset H_R$.

We now find an $\rho > 0$ such that

$$\Pi_{\rho:1}(x^0) \subset \Pi_{\frac{R}{4h}:1}(x^0).$$

For this, it is sufficient that the inequality

$$\omega_i^{-1}(\rho) = \omega_i^{-1}\left(\frac{R}{4h}:1\right)$$

holds for $i = 1, \ldots, n$.

By choosing $\rho = \frac{R}{4h}$, we ensure the validity of the required inclusion. On the other hand, from

$$cap_L^{(\Pi_{R:2}(0))}\left(\Pi_{\frac{R}{4h}:1}(x^0)\right) \geq$$

$$\geq c_{27}(\gamma, n) \cdot \left(\frac{R}{4h}\right)^{-2} \cdot \prod_{i=1}^{n} \omega_i^{-1}\left(\frac{R}{4h}\right).$$

Therefore, for $m = 1,2, \ldots$, we have

$$cap_L^{(\Pi_{A^m:2}(0))}(H^m) = cap_L^{(\Pi_{A^m:2}(0))}\left(\Pi_{\frac{A^m}{4h}:1}(x^0)\right) \geq$$

$$\geq c_{27} \cdot \left(\frac{A^m}{4h}\right)^{-2} \cdot \prod_{i=1}^{n} \omega_i^{-1}\left(\frac{A^m}{4h}\right). \tag{63}$$

Applying Theorem 12 now and taking into account (63), we obtain that if alternative 1) does not hold, then

$$\varliminf_{R\to\infty} M(R) \,/$$

$$\exp\left[c(\gamma, n) \cdot \sum_{i=1}^{[\ln R]} \left(\frac{A}{4h}\right)^{2m} \cdot \left(\prod_{j=1}^{n} \omega_i^{-1}\left(\frac{A^m}{4h}\right)\right)^{-1} \times\right.$$

$$\left. \times\, c_{27} \cdot \left(\frac{A^m}{4h}\right)^{-2} \cdot \prod_{i=1}^{n} \omega_i^{-1}\left(\frac{A^m}{4h}\right)\right] =$$

$$= \varliminf_{R\to\infty} M(R) \,/\, \exp\left[c(\gamma, n) \cdot c_{27} \cdot [\ln R]\right] =$$

$$= \varliminf_{R\to\infty} M(R) \,/\, \exp(\eta_1 \cdot [\ln R]) > 0.$$

Thus, Theorem 13 is proven with $\eta_1 = c(\gamma, n) \cdot c_{27}$

4. Conclusion

The analogue of the Phragmén–Lindelöf principle for the investigated class of non-uniformly degenerate equations asserts that the behavior of the solution at infinity is strictly determined by the local characteristic of the ellipticity degeneracy. Provided that the structural conditions on the weight functions w(x) are met, any solution with a growth rate below the critical exponential order inevitably obeys the maximum principle established for bounded domains. Furthermore, the loss of uniformity in degeneracy leads to a narrowing of the uniqueness class compared to the classical case.

References

1. Amanov R.A. Estimation of Green function of the first boundary value problem for the second order nonuniformly degenerated divergent elliptic equations, Proceedings of IMM of NAS of Azerbaijan., V.19, N.27, (2003), pp.15-35.
2. Aliguliyev R.M. A Phragmen-Lindelöf type theorem for a non-divergent structure linear elliptic equation of the second order, Trans. Nat. Acad. Sci. Azerb., V.21, N.4, (2001), pp.16-24.
3. Huseynov S.T. Qualitative properties of solutions of divergent elliptic equations of the second order with non-uniform power degeneration. Ph.D. thesis, Baku-2000.
4. Huseynov R.V. Phragmen-Lindelöf type theorem for solutions of quasi-elliptic equations, Proceedings of the Azerbaijan National Academy of Sciences, ser. on phy. and tec. sci., V.8, N.5, (1987), pp.29-31.
5. Kurta V.V. On Fragmen-Lindelöf's theorems for semi linear equations, Dok. AN SSSR, V.322, N.21, (1992), pp.32-40.
6. Kurta V.V. On Fragmen–Lindelöf's theorems for elliptic equations, Suc. Math. Sci., V.47, N.23, (1992), pp.165-186.
7. Landis E.M. S-capacity and its applications to the study of solutions of second-order elliptic equations with discontinuous coefficients, Math. Coll., V.292, N.4, (1968), pp.186-213.
8. Liang Xiting. A Phragmen-Lindelöf principle for generalized solutions of quasilinear elliptic equations, Northeast Math. J., V.5, N.2, (1988), pp.170-178.
9. Mamedov F.M. Phragmen-Lindelöf type theorem for linear divergent elliptic equations of the second order with divergent right-hand side, Diff. Equations, V.26, N.11, (1990), pp.1971-1978.
10. Mamedov I.T. On a generalization of the Phragmen-Lindelöf theorem for linear elliptic equations, Math. Notes., V.339, N.3, (1983), pp.357-364.
11. Mamedov I.T. An analogue of the Phragmen-Lindelöf principle for solutions of degenerate elliptic equations of the second order,

Proceedings of the Azerbaijan National Academy of Sciences, series p.m.s., V.18, N. 4-5, (1997), pp.13-122.